

# On the blow-up problem and new *a priori* estimates for the 3D Euler and the Navier-Stokes equations

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## Abstract

We study blow-up rates and the blow-up profiles of possible asymptotically self-similar singularities of the 3D Euler equations, where the sense of convergence and self-similarity are considered in various sense. We extend much further, in particular, the previous nonexistence results of self-similar/asymptotically self-similar singularities obtained in [2, 3]. Some implications the notions for the 3D Navier-Stokes equations are also deduced. Generalization of the self-similar transforms is also considered, and by appropriate choice of the transform we obtain new *a priori* estimates for the 3D Euler and the Navier-Stokes equations.

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# 1 Asymptotically self-similar singularities

We are concerned on the following Euler equations for the homogeneous incompressible fluid flows in  $\mathbb{R}^3$ .

$$(E) \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ \operatorname{div} v = 0, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^3 \end{cases}$$

where  $v = (v_1, v_2, v_3)$ ,  $v_j = v_j(x, t)$ ,  $j = 1, 2, 3$ , is the velocity of the flow,  $p = p(x, t)$  is the scalar pressure, and  $v_0$  is the given initial velocity, satisfying  $\operatorname{div} v_0 = 0$ . The system (E) is first modeled by Euler in [13]. The local well-posedness of the Euler equations in  $H^m(\mathbb{R}^3)$ ,  $m > 5/2$ , is established by Kato in [17], which says that given  $v_0 \in H^m(\mathbb{R}^3)$ , there exists  $T \in (0, \infty]$  such that there exists unique solution to (E),  $v \in C([0, T]; H^m(\mathbb{R}^3))$ . The finite time blow-up problem of the local classical solution is known as one of the most important and difficult problems in partial differential equations (see e.g. [20, 6, 7, 8, 2] for graduate level texts and survey articles on the current status of the problem). We say a local in time classical solution  $v \in C([0, T]; H^m(\mathbb{R}^3))$  blows up at  $T$  if  $\limsup_{t \rightarrow T} \|v(t)\|_{H^m} = \infty$  for all  $m > 5/2$ . The celebrated Beale-Kato-Majda criterion([1]) states that the blow-up happens at  $T$  if and only if

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = \infty.$$

There are studies of geometric nature for the blow-up criterion([9, 8, 12]). As another direction of studies of the blow-up problem mathematicians also consider various scenarios of singularities and study carefully their possibility of realization (see e.g. [10, 11, 3, 4] for some of those studies). One of the purposes in this paper, especially in this section, is to study more deeply the notions related to the scenarios of the self-similar singularities in the Euler equations, the preliminary studies of which are done in [3, 4]. We recall that system (E) has scaling property that if  $(v, p)$  is a solution of the system (E), then for any  $\lambda > 0$  and  $\alpha \in \mathbb{R}$  the functions

$$v^{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda, \alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \quad (1.1)$$

are also solutions of (E) with the initial data  $v_0^{\lambda, \alpha}(x) = \lambda^\alpha v_0(\lambda x)$ . In view of the scaling properties in (1.1), a natural self-similar blowing up solution

$v(x, t)$  of (E) should be of the form,

$$v(x, t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} \bar{V} \left( \frac{x}{(T-t)^{\frac{1}{\alpha+1}}} \right) \quad (1.2)$$

$$p(x, t) = \frac{\alpha+1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} \bar{P} \left( \frac{x}{(T-t)^{\frac{1}{\alpha+1}}} \right) \quad (1.3)$$

for  $\alpha \neq -1$  and  $t$  sufficiently close to  $T$ . Substituting (1.2)-(1.3) into (E), we obtain the following stationary system.

$$\begin{cases} \alpha \bar{V} + (y \cdot \nabla) \bar{V} + (\alpha+1)(\bar{V} \cdot \nabla) \bar{V} = -\nabla \bar{P}, \\ \operatorname{div} \bar{V} = 0, \end{cases} \quad (1.4)$$

the Navier-Stokes equations version of which has been studied extensively after Leray's pioneering paper([19, 23, 24, 22, 4, 16]). Existence of solution of the system (1.4) is equivalent to the existence of solutions to the Euler equations of the form (1.2)-(1.3), which blows up in a self-similar fashion. Given  $(\alpha, p) \in (-1, \infty) \times (0, \infty]$ , we say the blow-up is  $\alpha$ -asymptotically self-similar in the sense of  $L^p$  if there exists  $\bar{V} = \bar{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)$  such that the following convergence holds true.

$$\lim_{t \rightarrow T} (T-t) \left\| \nabla v(\cdot, t) - \frac{1}{T-t} \nabla \bar{V} \left( \frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{L^\infty} = 0$$

if  $p = \infty$ , while

$$\lim_{t \rightarrow T} (T-t)^{1-\frac{3}{(\alpha+1)p}} \left\| \omega(\cdot, t) - \frac{1}{(T-t)^{1-\frac{3}{(\alpha+1)p}}} \bar{\Omega} \left( \frac{\cdot}{(T-t)^{\frac{1}{\alpha+1}}} \right) \right\|_{L^p} = 0$$

if  $0 < p < \infty$ , where and hereafter we denote

$$\Omega = \operatorname{curl} V \quad \text{and} \quad \bar{\Omega} = \operatorname{curl} \bar{V}.$$

The above limit function  $\bar{V} \in L^p(\mathbb{R}^3)$  with  $\bar{\Omega} \neq 0$  is called the *blow-up profile*. We observe that the self-similar blow-up given by (1.2)-(1.3) is trivial case of  $\alpha$ -asymptotic self-similar blow-up with the blow-up profile given by the representing function  $\bar{V}$ . We say a blow-up at  $T$  is of *type I*, if

$$\limsup_{t \rightarrow T} (T-t) \|\nabla v(t)\|_{L^\infty} < \infty.$$

If the blow-up is not of type I, we say it is of *type II*. For the use of terminology, type I and type II blow-ups, we followed the literatures on the studies of the blow-up problem in the semilinear heat equations (see e.g. [21, 14, 15], and references therein). The use of  $\|\nabla v(t)\|_{L^\infty}$  rather than  $\|v(t)\|_{L^\infty}$  in our definition of type I and II is motivated by Beale-Kato-Majda's blow-up criterion.

**Theorem 1.1** *Let  $m > 5/2$ , and  $v \in C([0, T]; H^m(\mathbb{R}^3))$  be a solution to (E) with  $v_0 \in H^m(\mathbb{R}^3)$ ,  $\operatorname{div} v_0 = 0$ . We set*

$$\limsup_{t \rightarrow T} (T - t) \|\nabla v(t)\|_{L^\infty} := M(T). \quad (1.5)$$

*Then, either  $M(T) = 0$  or  $M(T) \geq 1$ . The former case corresponds to non blow-up, and the latter case corresponds to the blow-up at  $T$ . Hence, the blow-up at  $T$  is of type I if and only if  $M(T) \geq 1$ .*

**Proof** It suffices to show that  $M(T) < 1$  implies non blow-up at  $T$ , which, in turn, leads to  $M(T) = 0$ , since  $\|\nabla v(t)\|_{L^\infty} \in C([0, T])$  in this case. We suppose  $M(T) < 1$ . Then, there exists  $t_0 \in (0, T)$  such that

$$\sup_{t_0 < t < T} (T - t) \|\nabla v(t)\|_{L^\infty} := M_0 < 1.$$

Taking curl of the evolution part of (E), we have the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega = (\omega \cdot \nabla) v.$$

This, taking dot product with  $\xi = \omega/|\omega|$ , leads to

$$\frac{\partial |\omega|}{\partial t} + (v \cdot \nabla) |\omega| = (\xi \cdot \nabla) v \cdot \xi |\omega|.$$

Integrating this over  $[t_0, t]$  along the particle trajectories  $\{X(a, t)\}$  defined by  $v(x, t)$ , we have

$$|\omega(X(a, t), t)| = |\omega(X(a, t_0), t_0)| \exp \left[ \int_{t_0}^t (\xi \cdot \nabla) v \cdot \xi(X(a, s), s) ds \right], \quad (1.6)$$

from which we estimate

$$\begin{aligned}
\|\omega(t)\|_{L^\infty} &\leq \|\omega(t_0)\|_{L^\infty} \exp \left[ \int_{t_0}^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] \\
&< \|\omega(t_0)\|_{L^\infty} \exp \left[ M_0 \int_{t_0}^t (T - \tau)^{-1} d\tau \right] \\
&= \|\omega(t_0)\|_{L^\infty} \left( \frac{T - t_0}{T - t} \right)^{M_0}.
\end{aligned} \tag{1.7}$$

Since  $M_0 < 1$ , we have  $\int_{t_0}^T \|\omega(t)\|_{L^\infty} dt < \infty$ , and thanks to the Beale-Kato-Majda criterion there exists no blow-up at  $T$ , and we can continue our classical solution beyond  $T$ .  $\square$

The following is our main theorem in this section.

**Theorem 1.2** *Let a classical solution  $v \in C([0, T]; H^m(\mathbb{R}^3))$  with initial data  $v_0 \in H^m(\mathbb{R}^3) \cap \dot{W}^{1,p}(\mathbb{R}^3)$ ,  $\operatorname{div} v_0 = 0$ ,  $\omega_0 \neq 0$  blows up with type I. Let  $M = M(T)$  be as in Theorem 1.1. Suppose  $(\alpha, p) \in (-1, \infty) \times (0, \infty]$  satisfies*

$$M < \left| 1 - \frac{3}{(\alpha + 1)p} \right|. \tag{1.8}$$

*Then, there exists no  $\alpha$ -asymptotically self-similar blow-up at  $t = T$  in the sense of  $L^p$  if  $\omega_0 \in L^p(\mathbb{R}^3)$ . Hence, for any type I blow-up and for any  $\alpha \in (-1, \infty)$  there exists  $p_1 \in (0, \infty]$  such that it is not  $\alpha$ -asymptotically self-similar in the sense of  $L^{p_1}$ .*

**Remark 1.1** We note that the case  $p = \infty$  of the above theorem follows from Theorem 1.1, which states that there is no singularity at all at  $t = T$  in this case. The above theorem can be regarded an improvement of the main theorem in [4], in the sense that we can consider the  $L^p$  convergence only to exclude nontrivial blow-up profile  $\bar{V}$ , where  $p$  depends on  $M$ . Moreover, we do not need to use the Besov space  $\dot{B}_{\infty,1}^0$  in the statement of the theorem, and the continuation principle of local solution in the Besov space in the proof.

**Proof of Theorem 1.2** We assume asymptotically self-similar blow-up happens at  $T$ . Let us introduce similarity variables defined by

$$y = \frac{x}{(T - t)^{\frac{1}{\alpha+1}}}, \quad s = \frac{1}{\alpha + 1} \log \left( \frac{T}{T - t} \right),$$

and transformation of the unknowns  $(v, p) \rightarrow (V, P)$  according to

$$v(x, t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} V(y, s), \quad p(x, t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} P(y, s). \quad (1.9)$$

Substituting  $(v, p)$  into the (E) we obtain the equivalent evolution equation for  $(V, P)$ ,

$$(E_1) \begin{cases} V_s + \alpha V + (y \cdot \nabla) V + (\alpha + 1)(V \cdot \nabla) V = -\nabla P, \\ \operatorname{div} V = 0, \\ V(y, 0) = V_0(y) = T^{\frac{\alpha}{\alpha+1}} v_0(T^{\frac{1}{\alpha}} y). \end{cases}$$

Then the assumption of asymptotically self-similar singularity at  $T$  implies that there exists  $\bar{V} = \bar{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)$  such that

$$\lim_{s \rightarrow \infty} \|\Omega(\cdot, s) - \bar{\Omega}\|_{L^p} = 0. \quad (1.10)$$

Now the hypothesis (1.8) implies that there exists  $t_0 \in (0, T)$  such that

$$\sup_{t_0 < t < T} (T-t) \|\nabla v(t)\|_{L^\infty} := M_0 < \left| 1 - \frac{3}{(\alpha+1)p} \right|. \quad (1.11)$$

Taking  $L^p(\mathbb{R}^3)$  norm of (1.6), taking into account the following simple estimates,

$$-\|\nabla v(\cdot, t)\|_{L^\infty} \leq (\xi \cdot \nabla) v \cdot \xi(x, t) \leq \|\nabla v(\cdot, t)\|_{L^\infty} \quad \forall (x, t) \in \mathbb{R}^3 \times [t_0, T),$$

we obtain, for all  $p \in (0, \infty]$ ,

$$\begin{aligned} \|\omega(t_0)\|_{L^p} \exp \left[ - \int_{t_0}^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] &\leq \|\omega(t)\|_{L^p} \\ &\leq \|\omega_0\|_{L^p} \exp \left[ \int_{t_0}^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right], \end{aligned} \quad (1.12)$$

where we use the fact that  $a \mapsto X(a, t)$  is a volume preserving map. From the fact

$$\int_{t_0}^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \leq M_0 \int_{t_0}^t (T-\tau)^{-1} d\tau = -M_0 \log \left( \frac{T-t}{T-t_0} \right),$$

and

$$\frac{\|\omega(t)\|_{L^p}}{\|\omega(t_0)\|_{L^p}} = \left( \frac{T-t}{T-t_0} \right)^{\frac{3}{(\alpha+1)p}-1} \frac{\|\Omega(s)\|_{L^p}}{\|\Omega(s_0)\|_{L^p}},$$

where we set

$$s_0 = \frac{1}{\alpha+1} \log \left( \frac{T}{T-t_0} \right),$$

we find that (1.1) leads us to

$$\left( \frac{T-t}{T-t_0} \right)^{M_0+1-\frac{3}{(\alpha+1)p}} \leq \frac{\|\Omega(s)\|_{L^p}}{\|\Omega(s_0)\|_{L^p}} \leq \left( \frac{T-t}{T-t_0} \right)^{-M_0+1-\frac{3}{(\alpha+1)p}} \quad (1.13)$$

for all  $p \in (0, \infty]$ . Passing  $t \rightarrow T$ , which is equivalent to  $s \rightarrow \infty$  in (1.13), we have from (1.10)

$$\lim_{s \rightarrow \infty} \frac{\|\Omega(s)\|_{L^p}}{\|\Omega(s_0)\|_{L^p}} = \frac{\|\bar{\Omega}\|_{L^p}}{\|\Omega(s_0)\|_{L^p}} \in (0, \infty). \quad (1.14)$$

By (1.11)  $M_0 + 1 - \frac{3}{(\alpha+1)p} < 0$  or  $-M_0 + 1 - \frac{3}{(\alpha+1)p} > 0$ . In the former case we have

$$\lim_{t \rightarrow T} \left( \frac{T-t}{T-t_0} \right)^{M_0+1-\frac{3}{(\alpha+1)p}} = \infty, \quad (1.15)$$

while, in the latter case

$$\lim_{t \rightarrow T} \left( \frac{T-t}{T-t_0} \right)^{-M_0+1-\frac{3}{(\alpha+1)p}} = 0. \quad (1.16)$$

Both of (1.15) and (1.16) contradicts with (1.14). If the blow-up is of type I, and  $M(T) < \infty$ , then one can always choose  $p_1 \in (0, p_0)$  so small that (1.8) is valid for  $p = p_1$ . With such  $p_1$  it is not  $\alpha$ -asymptotically self-similar in  $L^{p_1}$ .  $\square$

For the self-similar blowing-up solution of the form (1.2)-(1.3) we observe that in order to be consistent with the energy conservation,  $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$  for all  $t \in [0, T)$ , we need to fix  $\alpha = 3/2$ . Since the self-similar blowing up solution corresponds to a trivial convergence of the asymptotically self-similar blow-up, the following is immediate from Theorem 1.2.

**Corollary 1.1** *Given  $p \in (0, \infty]$ , there exists no self-similar blow-up with the blow-up profile  $V$  satisfying  $\Omega \in L^p(\mathbb{R}^3)$  if*

$$\|\nabla V\|_{L^\infty} < \left|1 - \frac{6}{5p}\right|. \quad (1.17)$$

**Remark 1.2** The above corollary implies that we can exclude self-similar singularity of the Euler equations only under the assumption of  $\Omega \in L^p(\mathbb{R}^3)$  if  $p$  satisfies the condition (1.17).

The following is, in turn, immediate from the above corollary, which is nothing but Theorem 1.1 in [3].

**Corollary 1.2** *There exists no self-similar blow-up with the blow-up profile  $V$  satisfying  $\Omega \in L^p(\mathbb{R}^3)$  for all  $p \in (0, p_0)$  for some  $p_0 > 0$ .*

The following theorem is concerned on the possibility of type II asymptotically self-similar singularity of the Euler equations, for which the blow-up rate near the possible blow-up time  $T$  is

$$\|\nabla v(t)\|_{L^\infty} \sim \frac{1}{(T-t)^\gamma}, \quad \gamma > 1. \quad (1.18)$$

**Theorem 1.3** *Let  $v \in C([0, T); H^m(\mathbb{R}^3))$ ,  $m > 5/2$ , be local classical solution of the Euler equations. Suppose there exists  $\gamma > 1$  and  $R_1 > 0$  such that the following convergence holds true.*

$$\lim_{t \rightarrow T} (T-t)^{(\alpha-\frac{3}{2})\frac{\gamma}{\alpha+1}} \left\| v(\cdot, t) - \frac{1}{(T-t)^{(\alpha-\frac{3}{2})\frac{\gamma}{\alpha+1}}} \bar{V} \left( \frac{\cdot}{(T-t)^{\frac{\gamma}{\alpha+1}}} \right) \right\|_{L^2(B_{R_1})} = 0, \quad (1.19)$$

where  $B_{R_1} = \{x \in \mathbb{R}^3 \mid |x| < R_1\}$ . Then, the blow-up profile  $\bar{V} \in L^2_{loc}(\mathbb{R}^3)$  is a weak solution of the following stationary Euler equations,

$$(\bar{V} \cdot \nabla) \bar{V} = -\nabla \bar{P}, \quad \operatorname{div} \bar{V} = 0. \quad (1.20)$$

**Proof** We introduce a self-similar transform defined by

$$v(x, t) = \frac{1}{(T-t)^{\frac{\alpha\gamma}{\alpha+1}}} V(y, s), \quad p(x, t) = \frac{1}{(T-t)^{\frac{2\alpha\gamma}{\alpha+1}}} P(y, s) \quad (1.21)$$



with

$$y = \frac{1}{(T-t)^{\frac{\gamma}{\alpha+1}}}x, \quad s = \frac{1}{(\gamma-1)T^{\gamma-1}} \left[ \frac{T^{\gamma-1}}{(T-t)^{\gamma-1}} - 1 \right]. \quad (1.22)$$

Substituting  $(v, p)$  in (1.21)-(1.22) into the  $(E)$ , we have

$$(E_2) \begin{cases} -\frac{\gamma}{s(\gamma-1)+T^{1-\gamma}} \left[ \frac{\alpha}{\alpha+1}V + \frac{1}{\alpha+1}(y \cdot \nabla)V \right] = V_s + (V \cdot \nabla)V + \nabla P, \\ \operatorname{div} V = 0, \\ V(y, 0) = V_0(y) = v_0(y). \end{cases} \quad (1.23)$$

The hypothesis (1.19) is written as

$$\lim_{s \rightarrow \infty} \|V(\cdot, s) - \bar{V}(\cdot)\|_{L^2(B_{R(s)})} = 0, \quad R(s) = \left[ (\gamma-1)s + \frac{1}{T^{\gamma-1}} \right]^{\frac{\gamma}{(\alpha+1)(\gamma-1)}}, \quad (1.24)$$

which implies that

$$\lim_{s \rightarrow \infty} \|V(\cdot, s) - \bar{V}\|_{L^2(B_R)} = 0, \quad \forall R > 0, \quad (1.25)$$

where  $V(y, s)$  is defined by (1.21). Similarly to [16, 4], we consider the scalar test function  $\xi \in C_0^1(0, 1)$  with  $\int_0^1 \xi(s)ds \neq 0$ , and the vector test function  $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^1(\mathbb{R}^3)$  with  $\operatorname{div} \phi = 0$ .

We multiply the first equation of  $(E_2)$ , in the dot product, by  $\xi(s-n)\phi(y)$ , and integrate it over  $\mathbb{R}^3 \times [n, n+1]$ , and then we integrate by parts to obtain

$$\begin{aligned} & + \frac{\alpha}{\alpha+1} \int_0^1 \int_{\mathbb{R}^3} g(s+n) \xi(s) V(y, s+n) \cdot \phi(y) dy ds \\ & - \frac{1}{\alpha+1} \int_0^1 \int_{\mathbb{R}^3} g(s+n) \xi(s) V(y, s+n) \cdot (y \cdot \nabla) \phi(y) dy ds \\ & = \int_0^1 \int_{\mathbb{R}^3} \xi_s(s) \phi(y) \cdot V(y, s+n) dy ds \\ & + \int_0^1 \int_{\mathbb{R}^3} \xi(s) [V(y, s+n) \cdot (V(y, s+n) \cdot \nabla) \phi(y)] dy ds = 0, \end{aligned}$$

where we set

$$g(s) = \frac{\gamma}{s(\gamma-1)+T^{1-\gamma}}.$$

Passing to the limit  $n \rightarrow \infty$  in this equation, using the facts  $\int_0^1 \xi_s(s) ds = 0$ ,  $\int_0^1 \xi(s) ds \neq 0$ ,  $V(\cdot, s+n) \rightarrow \bar{V}$  in  $L^2_{\text{loc}}(\mathbb{R}^3)$ , and finally  $g(s+n) \rightarrow 0$ , we find that  $\bar{V} \in L^2_{\text{loc}}(\mathbb{R}^3)$  satisfies

$$\int_{\mathbb{R}^3} \bar{V} \cdot (\bar{V} \cdot \nabla) \phi(y) dy = 0$$

for all vector test function  $\phi \in C_0^1(\mathbb{R}^3)$  with  $\text{div } \phi = 0$ . On the other hand, we can pass  $s \rightarrow \infty$  directly in the weak formulation of the second equation of  $(E_2)$  to have

$$\int_{\mathbb{R}^3} \bar{V} \cdot \nabla \psi(y) dy = 0$$

for all scalar test function  $\psi \in C_0^1(\mathbb{R}^3)$ .  $\square$

## 2 Generalized similarity transforms and new a priori estimates

Let us consider a classical solution to (E)  $v \in C([0, T]; H^m(\mathbb{R}^3))$ ,  $m > 5/2$ , where we assume  $T \in (0, \infty]$  is the maximal time of existence of the classical solution. Let  $p(x, t)$  be the associated pressure. Let  $\mu(\cdot) \in C^1([0, T])$  be a scalar function such that  $\mu(t) > 0$  for all  $t \in [0, T)$  and  $\int_0^T \mu(t) dt = \infty$ . We transform from  $(v, p)$  to  $(V, P)$  according to the formula,

$$v(x, t) = \mu(t)^{\frac{\alpha}{\alpha+1}} V \left( \mu(t)^{\frac{1}{\alpha+1}} x, \int_0^t \mu(\sigma) d\sigma \right), \quad (2.1)$$

$$p(x, t) = \mu(t)^{\frac{2\alpha}{\alpha+1}} P \left( \mu(t)^{\frac{1}{\alpha+1}} x, \int_0^t \mu(\sigma) d\sigma \right), \quad (2.2)$$

where  $\alpha \in (-1, \infty)$  as previously. This means that the space-time variables are transformed from  $(x, t) \in \mathbb{R}^3 \times [0, T)$  into  $(y, s) \in \mathbb{R}^3 \times [0, \infty)$  as follows:

$$y = \mu(t)^{\frac{1}{\alpha+1}} x, \quad s = \int_0^t \mu(\sigma) d\sigma. \quad (2.3)$$

Substituting (2.1)-(2.3) into the Euler equations, we obtain the equivalent equations satisfied by  $(V, P)$

$$(E_*) \left\{ \begin{array}{l} -\frac{\mu'(t)}{\mu(t)^2} \left[ \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V + \nabla P, \\ \operatorname{div} V = 0, \\ V(y, 0) = V_0(y) = v_0(y). \end{array} \right.$$

We note that the special cases

$$\mu(t) = \frac{1}{T-t}, \quad \mu(t) = \frac{1}{(T-t)^\gamma}, \gamma > 1$$

are considered in the previous section. In this section we choose  $\mu(t) = \exp \left[ \pm \gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]$ ,  $\gamma \geq 1$ . Then,

$$v(x, t) = \exp \left[ \frac{\pm \gamma \alpha}{\alpha+1} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] V(y, s), \quad (2.4)$$

$$p(x, t) = \exp \left[ \frac{\pm 2\gamma \alpha}{\alpha+1} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] P(y, s) \quad (2.5)$$

with

$$\begin{aligned} y &= \exp \left[ \frac{\pm \gamma}{\alpha+1} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] x, \\ s &= \int_0^t \exp \left[ \pm \gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \end{aligned} \quad (2.6)$$

respectively for the signs  $\pm$ . Substituting  $(v, p)$  in (2.4)-(2.6) into the  $(E)$ , we find that  $(E_*)$  becomes

$$(E_\pm) \left\{ \begin{array}{l} \mp \gamma \|\nabla V(s)\|_{L^\infty} \left[ \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} (y \cdot \nabla) V \right] = V_s + (V \cdot \nabla) V + \nabla P, \\ \operatorname{div} V = 0, \\ V(y, 0) = V_0(y) = v_0(y) \end{array} \right.$$

respectively for  $\pm$ . Similar equations to the system  $(E_\pm)$ , without the term involving  $(y \cdot \nabla) V$  are introduced and studied in [5], where similarity type of transform with respect to only time variables was considered. The argument

of the global/local well-posedness of the system  $(E_{\pm})$  respectively from the local well-posedness result of the Euler equations is as follows. We define

$$S^{\pm} = \int_0^T \exp \left[ \pm \gamma \int_0^{\tau} \|\nabla v(\sigma)\|_{L^{\infty}} d\sigma \right] d\tau.$$

Then,  $S^{\pm}$  is the maximal time of existence of classical solution for the system  $(E_{\pm})$ . We also note the following integral invariant of the transform,

$$\int_0^T \|\nabla v(t)\|_{L^{\infty}} dt = \int_0^{S^{\pm}} \|\nabla V^{\pm}(s)\|_{L^{\infty}} ds.$$

The key advantage of our choice of the function  $\mu(t)$  here is that the convection term is dominated by  $\mp \gamma \|\nabla V(s)\|_{L^{\infty}} V$  in the transformed system  $(E_{\pm})$  in the vorticity formulation, which enable us to derive new *a priori* estimates for  $\|\omega(t)\|_{L^{\infty}}$  as follows.

**Theorem 2.1** *Given  $m > 5/2$  and  $v_0 \in H^m(\mathbb{R}^3)$  with  $\operatorname{div} v_0 = 0$ , let  $\omega$  be the vorticity of the solution  $v \in C([0, T]; H^m(\mathbb{R}^3))$  to the Euler equations (E). Then we have an upper estimate*

$$\|\omega(t)\|_{L^{\infty}} \leq \frac{\|\omega_0\|_{L^{\infty}} \exp \left[ \gamma \int_0^t \|\nabla v(\tau)\|_{L^{\infty}} d\tau \right]}{1 + (\gamma - 1) \|\omega_0\|_{L^{\infty}} \int_0^t \exp \left[ \gamma \int_0^{\tau} \|\nabla v(\sigma)\|_{L^{\infty}} d\sigma \right] d\tau}, \quad (2.7)$$

and lower one

$$\|\omega(t)\|_{L^{\infty}} \geq \frac{\|\omega_0\|_{L^{\infty}} \exp \left[ -\gamma \int_0^t \|\nabla v(\tau)\|_{L^{\infty}} d\tau \right]}{1 - (\gamma - 1) \|\omega_0\|_{L^{\infty}} \int_0^t \exp \left[ -\gamma \int_0^{\tau} \|\nabla v(\sigma)\|_{L^{\infty}} d\sigma \right] d\tau} \quad (2.8)$$

for all  $\gamma \geq 1$  and  $t \in [0, T]$ . The denominator of the right hand side of (2.8) can be estimated from below as

$$1 - (\gamma - 1) \|\omega_0\|_{L^{\infty}} \int_0^t \exp \left[ -\gamma \int_0^{\tau} \|\nabla v(\sigma)\|_{L^{\infty}} d\sigma \right] d\tau \geq \frac{1}{(1 + \|\omega_0\|_{L^{\infty}} t)^{\gamma-1}}, \quad (2.9)$$

which shows that the finite time blow-up does not follow from (2.8).

**Remark 2.1** We observe that for  $\gamma = 1$ , the estimates (2.7)-(2.8) reduce to the well-known ones in (1.12) with  $p = \infty$ . Moreover, combining (2.7)-(2.8)

together, we easily derive another new estimate,

$$\frac{\sinh \left[ \gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]}{\int_0^t \cosh \left[ \gamma \int_\tau^t \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau} \geq (\gamma - 1) \|\omega_0\|_{L^\infty}. \quad (2.10)$$

**Proof of Theorem 2.1** Below we denote  $V^\pm$  for the solutions of  $(E_\pm)$  respectively, and  $\Omega^\pm = \text{curl } V^\pm$ . Note that  $V_0^\pm = v_0 := V_0$  and  $\Omega_0^\pm = \omega_0 := \Omega_0$ . We will first derive the following estimates for the system  $(E_\pm)$ .

$$\|\Omega^+(s)\|_{L^\infty} \leq \frac{\|\Omega_0\|_{L^\infty}}{1 + (\gamma - 1)s\|\Omega_0\|_{L^\infty}}, \quad (2.11)$$

$$\|\Omega^-(s)\|_{L^\infty} \geq \frac{\|\Omega_0\|_{L^\infty}}{1 - (\gamma - 1)s\|\Omega_0\|_{L^\infty}}, \quad (2.12)$$

as long as  $V^\pm(s) \in H^m(\mathbb{R}^3)$ . Taking curl of the first equation of  $(E_\pm)$ , we have

$$\mp \gamma \|\nabla V\|_{L^\infty} \left[ \Omega - \frac{1}{\alpha + 1} (y \cdot \nabla) \Omega \right] = \Omega_s + (V \cdot \nabla) \Omega - (\Omega \cdot \nabla) V. \quad (2.13)$$

Multiplying  $\Xi = \Omega/|\Omega|$  on the both sides of (2.13), we deduce

$$\begin{aligned} |\Omega|_s + (V \cdot \nabla) |\Omega| \mp \frac{\|\nabla V(s)\|_{L^\infty}}{\alpha + 1} (y \cdot \nabla) |\Omega| &= (\Xi \cdot \nabla V \cdot \Xi \mp \|\nabla V\|_{L^\infty}) |\Omega| \\ &\quad \mp (\gamma - 1) \|\nabla V\|_{L^\infty} |\Omega| \\ &\begin{cases} \leq -(\gamma - 1) \|\nabla V\|_{L^\infty} |\Omega| & \text{for } (E_+) \\ \geq (\gamma - 1) \|\nabla V\|_{L^\infty} |\Omega| & \text{for } (E_-), \end{cases} \end{aligned} \quad (2.14)$$

since  $|\Xi \cdot \nabla V \cdot \Xi| \leq |\nabla V| \leq \|\nabla V\|_{L^\infty}$ . Given smooth solution  $V(y, s)$  of  $(E_\pm)$ , we introduce the particle trajectories  $\{Y_\pm(a, s)\}$  defined by

$$\frac{\partial Y(a, s)}{\partial s} = V_\pm(Y(a, s), s) \mp \frac{\|\nabla V(s)\|_{L^\infty}}{\alpha + 1} Y(a, s) \quad ; \quad Y(a, 0) = a.$$

Recalling the estimate

$$\|\nabla V(s)\|_{L^\infty} \geq \|\Omega(s)\|_{L^\infty} \geq |\Omega(y, s)| \quad \forall y \in \mathbb{R}^3,$$

we can further estimate from (2.14)

$$\frac{\partial}{\partial s} |\Omega(Y(a, s), s)| \begin{cases} \leq -(\gamma - 1) |\Omega(Y(a, s), s)|^2 & \text{for } (E_+) \\ \geq (\gamma - 1) |\Omega(Y(a, s), s)|^2 & \text{for } (E_-). \end{cases} \quad (2.15)$$

Solving these differential inequalities (2.15) along the particle trajectories, we obtain that

$$|\Omega(Y(a, s), s)| \begin{cases} \leq \frac{|\Omega_0(a)|}{1 + (\gamma - 1)s|\Omega_0(a)|} & \text{for } (E_+) \\ \geq \frac{|\Omega_0(a)|}{1 - (\gamma - 1)s|\Omega_0(a)|} & \text{for } (E_-). \end{cases} \quad (2.16)$$

Writing the first inequality of (2.16) as

$$|\Omega^+(Y(a, s), s)| \leq \frac{1}{\frac{1}{|\Omega_0(a)|} + (\gamma - 1)s} \leq \frac{1}{\frac{1}{\|\Omega_0\|_{L^\infty}} + (\gamma - 1)s},$$

and then taking supremum over  $a \in \mathbb{R}^3$ , which is equivalent to taking supremum over  $Y(a, s) \in \mathbb{R}^3$  due to the fact that the mapping  $a \mapsto Y(a, s)$  is a deffeomorphism(although not volume preserving) on  $\mathbb{R}^3$  as long as  $V \in C([0, S]; H^m(\mathbb{R}^3))$ , we obtain (2.11). In order to derive (2.12) from the second inequality of (2.16), we first write

$$\|\Omega^-(s)\|_{L^\infty} \geq |\Omega(Y(a, s), s)| \geq \frac{1}{\frac{1}{|\Omega_0(a)|} - (\gamma - 1)s},$$

and than take supremum over  $a \in \mathbb{R}^3$ . Finally, in order to obtain (2.7)-(2.8), we just change variables from (2.11)-(2.12) back to the original physical ones, using the fact

$$\begin{aligned} \Omega^+(y, s) &= \exp \left[ -\gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] \omega(x, t), \\ s &= \int_0^t \exp \left[ \gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \end{aligned}$$

for (2.7), while in order to deduce (2.8) from (2.12) we substitute

$$\begin{aligned} \Omega^-(y, s) &= \exp \left[ \gamma \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right] \omega(x, t), \\ s &= \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau. \end{aligned}$$

Now we can rewrite (2.8) as

$$\|\omega(t)\|_{L^\infty} \geq -\frac{1}{\gamma-1} \frac{d}{dt} \log \left\{ 1 - (\gamma-1) \|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \right\}.$$

Thus,

$$\begin{aligned} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau &\geq \int_0^t \|\omega(\tau)\|_{L^\infty} d\tau \geq \\ &\geq -\frac{1}{\gamma-1} \log \left\{ 1 - (\gamma-1) \|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau \right\}. \end{aligned} \quad (2.17)$$

Setting

$$y(t) := 1 - (\gamma-1) \|\omega_0\|_{L^\infty} \int_0^t \exp \left[ -\gamma \int_0^\tau \|\nabla v(\sigma)\|_{L^\infty} d\sigma \right] d\tau,$$

We find further integrable structure in (2.17), which is

$$y'(t) \geq -(\gamma-1) \|\omega_0\|_{L^\infty} y(t)^{\frac{\gamma}{\gamma-1}}.$$

Solving this differential inequality, we obtain (2.9).  $\square$

*In the last part of this section we fix  $\mu(t) := \exp \left[ \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]$ .*

We assume our local classical solution in  $H^m(\mathbb{R}^3)$  blows up at  $T$ , and hence  $\mu(T-0) = \exp \left[ \int_0^T \|\nabla v(\tau)\|_{L^\infty} d\tau \right] = \infty$ . Given  $(\alpha, p) \in (-1, \infty) \times (0, \infty)$ , as previously, we say the blow-up is  $\alpha$ -asymptotically self-similar in the sense of  $L^p$  if there exists  $\bar{V} = \bar{V}_\alpha \in \dot{W}^{1,p}(\mathbb{R}^3)$  such that the following convergence holds true.

$$\lim_{t \rightarrow T} \mu(t)^{-1} \left\| \nabla v(\cdot, t) - \mu(t) \nabla \bar{V} \left( \mu(t)^{\frac{1}{\alpha+1}}(\cdot) \right) \right\|_{L^\infty} = 0 \quad (2.18)$$

for  $p = \infty$ , and

$$\lim_{t \rightarrow T} \mu(t)^{-1 + \frac{3}{(\alpha+1)p}} \left\| \omega(\cdot, t) - \mu(t)^{1 - \frac{3}{(\alpha+1)p}} \bar{\Omega} \left( \mu(t)^{\frac{1}{\alpha+1}}(\cdot) \right) \right\|_{L^p} = 0 \quad (2.19)$$

for  $p \in (0, \infty)$ . The above limiting function  $\bar{V}$  with  $\bar{\Omega} \neq 0$  is called the blow-up profile as previously.

**Proposition 2.1** *Let  $\alpha \neq 3/2$ . Then there exists no  $\alpha$ -asymptotically self-similar blow-up in the sense of  $L^\infty$  with the blow-up profile belongs to  $L^2(\mathbb{R}^3)$ .*

**Proof** Let us suppose that there exists  $\bar{V} \in \dot{W}^{1,\infty}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$  such that (2.18) holds, then we will show that  $\bar{V} = 0$ . In terms of the self-similar variables (2.18) is translated into

$$\lim_{s \rightarrow \infty} \|\nabla V(\cdot, s) - \nabla \bar{V}\|_{L^\infty} = 0,$$

where  $V$  is defined in (2.1). If  $\|\nabla \bar{V}\|_{L^\infty} = 0$ , then, the condition  $\bar{V} \in L^2(\mathbb{R}^3)$  implies that  $\bar{V} = 0$ , and there is nothing to prove. Let us suppose  $\|\nabla \bar{V}\|_{L^\infty} > 0$ . The equations satisfied  $\bar{V}$  are

$$\begin{cases} -\|\nabla \bar{V}\|_{L^\infty} \left[ \frac{\alpha}{\alpha+1} \bar{V} + \frac{1}{\alpha+1} (y \cdot \nabla) \bar{V} \right] = (\bar{V} \cdot \nabla) \bar{V} + \nabla \bar{P}, \\ \operatorname{div} \bar{V} = 0 \end{cases} \quad (2.20)$$

for a scalar function  $\bar{P}$ . Taking  $L^2(\mathbb{R}^3)$  inner product of the first equation of (2.20) by  $\bar{V}$  we obtain

$$\frac{\|\nabla \bar{V}\|_{L^\infty}}{\alpha+1} \left( \alpha - \frac{3}{2} \right) \|\bar{V}\|_{L^2} = 0.$$

Since  $\|\nabla \bar{V}\|_{L^\infty} \neq 0$  and  $\alpha \neq \frac{3}{2}$ , we have  $\|\bar{V}\|_{L^2} = 0$ , and  $\bar{V} = 0$ .  $\square$

**Proposition 2.2** *There exists no  $\alpha$ -asymptotically self-similar blowing up solution to (E) in the sense of  $L^p$  if  $0 < p < \frac{3}{2(\alpha+1)}$ .*

**Proof** Suppose there exists  $\alpha$ -asymptotically self-similar blow-up at  $T$  in the sense of  $L^p$ . Then, there exists  $\bar{\Omega} \in L^p(\mathbb{R}^3)$  such that, in terms of the self-similar variables introduced in (2.1)-(2.2), we have

$$\lim_{s \rightarrow \infty} \|\Omega(s)\|_{L^p} = \|\bar{\Omega}\|_{L^p} < \infty. \quad (2.21)$$

We represent the  $L^p$  norm of  $\|\omega(t)\|_{L^p}$  in terms of similarity variables to obtain

$$\|\omega(t)\|_{L^p} = \mu(t)^{1-\frac{3}{(\alpha+1)p}} \|\Omega(s)\|_{L^p}, \quad \mu(t) = \exp \left[ \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right]. \quad (2.22)$$



Substituting this into the lower estimate part of (1.12), we have

$$\mu(t)^{-2+\frac{3}{(\alpha+1)p}} \leq \frac{\|\Omega(s)\|_{L^p}}{\|\Omega_0\|_{L^p}}. \quad (2.23)$$

If  $-2 + \frac{3}{(\alpha+1)p} > 0$ , then taking  $t \rightarrow T$  the above inequality we obtain,

$$\begin{aligned} \infty &= \limsup_{t \rightarrow T} \mu(t)^{-2+\frac{3}{(\alpha+1)p}} \|\Omega_0\|_{L^p} \\ &\leq \limsup_{s \rightarrow \infty} \|\Omega(s)\|_{L^p} = \|\bar{\Omega}\|_{L^p}, \end{aligned}$$

which is a contradiction to (2.21).  $\square$

### 3 The case of the 3D Navier-Stokes equations

In this section we concentrate on the following 3D Navier-Stokes equations in  $\mathbb{R}^3$  without forcing term.

$$(NS) \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = \Delta v - \nabla p, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ \operatorname{div} v = 0, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ v(x, 0) = v_0(x) & x \in \mathbb{R}^3. \end{cases}$$

First, we exclude asymptotically self-similar singularity of type II of (NS), for which the blow-up rate is given by (1.18). We have the following theorem.

**Theorem 3.1** *Let  $p \in [3, \infty)$  and  $v \in C([0, T]; L^p(\mathbb{R}^3))$  be a local classical solution of the Navier-Stokes equations constructed by Kato([18]). Suppose there exists  $\gamma > 1$  and  $\bar{V} \in L^p(\mathbb{R}^3)$  such that the following convergence holds true.*

$$\lim_{t \rightarrow T} (T - t)^{\frac{(p-3)\gamma}{2p}} \left\| v(\cdot, t) - (T - t)^{-\frac{(p-3)\gamma}{2p}} \bar{V} \left( \frac{\cdot}{(T - t)^{\frac{\gamma}{2}}} \right) \right\|_{L^p} = 0, \quad (3.1)$$

*If the blow-up profile  $\bar{V}$  belongs to  $\dot{H}^1(\mathbb{R}^3)$ , then  $\bar{V} = 0$ .*

**Proof** Since the main part of the proof is essentially identical to that of Theorem 1.3, we will be brief. Introducing the self-similar variables of the form (1.21)-(1.23) with  $\alpha = \frac{1}{2}$ , and substituting  $(v, p)$  into the Navier-Stokes equations, we find that  $(V, P)$  satisfies

$$\begin{cases} -\frac{\gamma}{2s(\gamma-1)+2T^{1-\gamma}}[V+(y\cdot\nabla)V]=V_s+(V\cdot\nabla)V-\Delta V+\nabla P, \\ \operatorname{div} V=0, \\ V(y,0)=V_0(y)=v_0(y). \end{cases}$$

The hypothesis (3.1) is now translated as

$$\lim_{s\rightarrow\infty}\|V(\cdot,s)-\bar{V}(\cdot)\|_{L^p}=0$$

Following exactly same argument as in the proof of Theorem 1.3, we can deduce that  $\bar{V}$  is a stationary solution of the Navier-Stokes equations, namely there exists  $\bar{P}$  such that

$$(\bar{V}\cdot\nabla)\bar{V}=\Delta\bar{V}-\nabla\bar{P}, \quad \operatorname{div}\bar{V}=0. \quad (3.2)$$

In the case  $\bar{V}\in\dot{H}^1\cap L^p(\mathbb{R}^3)$ , we easily from (3.2) that  $\int_{\mathbb{R}^3}|\nabla\bar{V}|^2dy=0$ , which implies  $\bar{V}=0$ .  $\square$

Next, we derive a new a priori estimates for classical solutions of the 3D Navier-stokes equations.

**Theorem 3.2** *Given  $v_0\in H^1(\mathbb{R}^3)$  with  $\operatorname{div} v_0=0$ , let  $\omega$  be the vorticity of the classical solution  $v\in C([0,T];H^1(\mathbb{R}^3))\cap C((0,T);C^\infty(\mathbb{R}^3))$  to the Navier-Stokes equations (NS). Then, there exists an absolute constant  $C_0>1$  such that for all  $\gamma\geq C_0$  the following enstrophy estimate holds true.*

$$\|\omega(t)\|_{L^2}\leq\frac{\|\omega_0\|_{L^2}\exp\left[\frac{\gamma}{4}\int_0^t\|\omega(\tau)\|_{L^2}^4d\tau\right]}{\left\{1+(\gamma-C_0)\|\omega_0\|_{L^2}^4\int_0^t\exp\left[\gamma\int_0^\tau\|\omega(\sigma)\|_{L^2}^4d\sigma\right]d\tau\right\}^{\frac{1}{4}}}. \quad (3.3)$$

The denominator of (3.3) is estimated from below by

$$1+(\gamma-C_0)\|\omega_0\|_{L^2}^4\int_0^t\exp\left[\gamma\int_0^\tau\|\omega(\sigma)\|_{L^2}^4d\sigma\right]d\tau\leq\frac{1}{(1-C_0\|\omega_0\|_{L^2}^4t)^{\frac{\gamma-C_0}{C_0}}} \quad (3.4)$$

for all  $\gamma\geq C_0$ .

**Proof** Let  $(v, p)$  be a classical solution of the Navier-Stokes equations, and  $\omega$  be its vorticity. We transform from  $(v, p)$  to  $(V, P)$  according to the formula, given by (2.1)-(2.3), where

$$\mu(t) = \exp \left[ \gamma \int_0^t \|\omega(\tau)\|_{L^2}^4 d\tau \right].$$

Substituting (2.1)-(2.3) with such  $\mu(t)$  into (NS), we obtain the equivalent equations satisfied by  $(V, P)$

$$(NS_*) \begin{cases} \frac{-\gamma \|\Omega(s)\|_{L^2}^4}{2} [V + (y \cdot \nabla)V] = V_s + (V \cdot \nabla)V - \Delta V - \nabla P, \\ \operatorname{div} V = 0, \\ V(y, 0) = V_0(y) = v_0(y). \end{cases}$$

Operating curl on the evolution equations of  $(NS_*)$ , we obtain

$$\frac{-\gamma \|\Omega(s)\|_{L^2}^4}{2} [2\Omega + (y \cdot \nabla)\Omega] = \Omega_s + (V \cdot \nabla)\Omega - (\Omega \cdot \nabla)V - \Delta\Omega. \quad (3.5)$$

Taking  $L^2(\mathbb{R}^3)$  inner product of (3.5) by  $\Omega$ , and integrating by part, we estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 + \frac{\gamma}{4} \|\Omega\|_{L^2}^6 &= \int_{\mathbb{R}^3} (\Omega \cdot \nabla)V \cdot \Omega dy \\ &\leq \|\Omega\|_{L^3} \|\nabla V\|_{L^2} \|\Omega\|_{L^6} \leq C \|\Omega\|_{L^2}^{\frac{3}{2}} \|\nabla \Omega\|_{L^2}^{\frac{3}{2}} \\ &\leq \|\nabla \Omega\|_{L^2}^2 + \frac{C_0}{4} \|\Omega\|_{L^2}^6 \end{aligned} \quad (3.6)$$

for an absolute constant  $C_0 > 1$ , where we used the fact  $\|\Omega\|_{L^2} = \|\nabla V\|_{L^2}$ , the Sobolev imbedding,  $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ , the Gagliardo-Nirenberg inequality in  $\mathbb{R}^3$ ,

$$\|f\|_{L^3} \leq C \|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}}.$$

and Young's inequality of the form  $ab \leq a^p/p + b^q/q$ ,  $1/p + 1/q = 1$ . Absorbing the term  $\|\nabla \Omega\|_{L^2}^2$  to the left hand side, we have from (3.6)

$$\frac{d}{ds} \|\Omega\|_{L^2}^2 \leq -\frac{\gamma - C_0}{2} \|\Omega\|_{L^2}^6. \quad (3.7)$$

Solving the differential inequality (3.7), we have

$$\|\Omega(s)\|_{L^2} \leq \frac{\|\Omega_0\|_{L^2}}{[1 + (\gamma - C_0)s\|\Omega_0\|_{L^2}^4]^{\frac{1}{4}}}. \quad (3.8)$$

Transforming back to the original variables and functions, using the relations

$$\begin{aligned} s &= \int_0^t \exp \left[ \gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma \right] d\tau, \\ \|\omega(t)\|_{L^2} &= \|\Omega(s)\|_{L^2} \exp \left[ \frac{\gamma}{4} \int_0^t \|\omega(\tau)\|_{L^2}^4 d\tau \right], \end{aligned}$$

we obtain (3.3). Next, we observe (3.3) can be written as

$$\|\omega(t)\|_{L^2}^4 \leq \frac{1}{(\gamma - C_0)} \frac{d}{dt} \log \left\{ 1 + (\gamma - C_0)\|\omega_0\|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma \right] d\tau \right\},$$

which, after integration over  $[0, t]$ , leads to

$$\int_0^t \|\omega(\tau)\|_{L^2}^4 d\tau \leq \frac{1}{(\gamma - C_0)} \log \left\{ 1 + (\gamma - C_0)\|\omega_0\|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma \right] d\tau \right\} \quad (3.9)$$

for all  $\gamma > C_0$ . Setting

$$y(t) := 1 + (\gamma - C_0)\|\omega_0\|_{L^2}^4 \int_0^t \exp \left[ \gamma \int_0^\tau \|\omega(\sigma)\|_{L^2}^4 d\sigma \right] d\tau,$$

we find that (3.9) can be written in the form of a differential inequality,

$$y'(t) \leq (\gamma - C_0)\|\omega_0\|_{L^2}^4 y(t)^{\frac{\gamma}{\gamma - C_0}},$$

which can be solved to provide us with (3.4).  $\square$

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